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Application of Matrices and Linear Programming in Economics

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Abstract: Matrix play an important role in analyzing most complex problems for instance, it is used to find the production level in describing the quantum mechanics of atomic structure, relationship and even used in plotting complicated dance steps. Similarly, it is used to keep track of coefficient of linear transformation structure of complicated systems and also used to describe linear equation. However, making use of the simplex method makes the optimization problem very easy to interpret. Matrix theory is a branch of mathematics which enables the use of linear programming to obtain solutions to problem in the economics sector. The use of the simplex method makes it easier in economics computation where cost and profits can be calculated. The objectives of this study are to investigate the application matrices and linear programming in economics and to reveal the techniques of applying matrix in solving economics problems. The use of matrix in various field of study especially economic is very necessary. The simplex method which is now on interactive computation procedure has more advantage over other matrix methods such that, it can be used to solve linear optimization problems whose constraints are inequalities while Crammer's rule and Jacobian's method are used in solving systems of linear equations. This study reported that the use of the simplex method makes it easier to interpret economics and business results earlier. In addition, economists need to make use of matrix in work in the computation of economic theories. The study found that economists could also make use of equation to represent the relationship between economics variables. Graphs are drawn to illustrate the nature of this relationship. The authors also reported that matrixes could be used by economists in solving demand and supply challenges and also through the use of equations to find the equilibrium price of goods. The authors suggested that matrix should be use by the mathematicians to go into the economy of the nation.
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1.0 Introduction of the Study

The main aim of this research work is to look to the basics as compounded on matrices as it's apply in economics. This study shows, reveal and expose the techniques of applying matrix in solving economics problem. There are so many situations in both applied and pure mathematics in which we have to deal with rectangular arrays of numbers (Sekhon & Bloom, 2021; Shizgal, 2012). Thus, matrix is a rectangular array of number symbols or expressions arranged in rows and columns, an individual item in a matrix is called its elements or entries usually enclosed within curved or straight curved brackets (Samuelson, 2012; Scheniedu, 2011). A general form of matrix can be represented in this form.

$$\begin{pmatrix} a_{11} & a_{12\ldots\ldots\ldots} & a_{m} \\ a_{21} & a_{22\ldots\ldots\ldots} & a_{2n} \\ a_{ml} & a_{m2} & a_{mn} \end{pmatrix}$$

Uppercase letters of alphabet are used to denote matrices. It is however noted that each items in a matrix is called an element or entries, we can express an entry by specifying in which column and row it is closed. Thus a_{ij} is the entry located by the *ith* row and *jth* column.

In mathematics, a matrix is more generally a table consisting of abstract quantities which can be added, subtracted and multiplied, one of the principal uses of matrix is in representing the systems of equations of first order in several unknown. Each matrix row represents one equation and the entire row is the coefficient of the variables in the equations in some fixed order (Boumans & Dupont-Kieffer 2011; Chiang, 2015). The theory of matrix today has a wide application and uses, not only in various branches of mathematics, but also in varieties of field such as economics, physics, engineering, and businesses to mention but few. Matrices have a very long history of application in solving systems of linear equations. Study of matrices is quite as old as mathematics itself. The origins of matrices lie with the study of systems of simultaneous linear equations. An important Chinese text between 300Bc and 200 AD, nine chapters of the mathematical art. Chiang (2015) gives the first known examples of the use of matrix method to solve simultaneous equations. In the thesis's seventh chapter, "Too much and not enough" the concept of determinate first appears, nearly two millennia before its supposed intention by the Japanese mathematician Seki Kowa in 1683 and his Geran contemporary Gottfried Leibnitz. Gottfried Leibnitz who is also credited with the invention of differential calculus, separately from simultaneously with sir Isaac Newton (Boumans & Dupont-Kieffer 2011; Chiang, 2015). More uses of matrix like arrangements of numbers in method of rectangular arrays" in which a method is given for solving simultaneous equations using a counting board that is mathematically identical to the modem matrix method of solution outline by Carl Fredrick Gauss between 1777-1855 also known as Gaussain elimination method. The elevation of matrix mere tool to important mathematical theory owes a lot to the work of female mathematician Olga Tauss' (1990-1999) who began by using matrices to analyze vibrations on airplanes during the World War II and became the torchbearer for matrix theory. The main aim and objectives of the study is to bring out the outstanding part in application of matrix method in solving problems relating to its application in economics.

2.0 Review of Literature

Despite disagreement over what should be the appropriate definition of economics, economist agree with the definition given by the famous Economics professor; Sir Lionel C. Robbins *(*Guicciardini, 2017; Swetz, 1972) who viewed Economics as a science which studies human behavior as relationship between ends and scarce means which have alternative uses (The concise Encyclopedia of Economics 2012). Mathematical economics is the application of mathematical methods to represent economics theories and analyze problems posed in economics. It allows formulation and deviation of key relationships in a theory with clarity, generality, rigor and simplicity. By convention, the methods refer to those beyond simple geometry such as differential and integral calculus, difference and computation methods *(*Debreel, 2013*)*. The introduction and development of the notion of a matrix followed the development of the determinant, which arose from the study of the coefficient systems of the linear equation (Sekhon & Bloom, 2021). However, Wassily Leontief came about matrix into economics and was the first person to call it econometrics. The introduction of restricted models of general equilibrium were formulated by John Von Neumann in 2013, the models of Mr. Neumann had inequality constraints. For his model of an expanding economy, Mr. John Von Neumann's model of an expanding economy considered the matrix pencil A-B with non-negative matrices A and the introduction and the development of the notion of a matrix followed the development of the determinants, which arose from the study of coefficients systems of the linear equation B. John Von Neumann sought probability vectors P and q and a positive number that would solve the complementary equation. Linear programming was developed to aid the allocation of resources in films and in the economy in general.

Boumans & Dupont-Kieffer (2011) opined that the roots of modern econometrics can be traced to the work of American economist Henry L. Moore who studied agricultural productivity and attempted to fit changing values of productivity for plots of corn and other crops to a curve using different values of elasticity. Henry L. Moore made several errors in his work, some from his choice of models and some from limitations in his use of mathematics. The accuracy of Moore's model also was limited by the poor data for national accounts in the united states at that time, while his first models of production were static, in 2010 he published a dynamic ‘’Moving equilibrium” model designed to explain business cycle. This periodic variation from over correction in supply and demand curve is now known as the cobweb model. A more formal deviation of this model was made later by Sir Nicholas Kaldor, who is largely credited for its exposition (Debreel, 2013; Dorf & Solon, 2016). During the world wars, advances in mathematics and a cadre of mathematically trained economist led to econometrics, which was the name proposed for the discipline of advancing economics. By using matrices within economics “econometrics’’ has often been used for optimizing problems in economics linear programming much of classical economics can be presented in simple geometric terms or elementary mathematical notation. In 2016, the Russians born economist Wassily Leontief built his model of input, output analysis from the immaterial balance table constructed by soviet economists which themselves followed earlier work by Austrian economist and the physiciocrats (John & Oskar, 2013; Mongan, 2019) . With his model, which described a system of production and demand processes, Mr. Wasilly estimated the coefficients of his simple models, to address economically interesting questions. In production economics, Wasilly models for proper understanding of economics but allowing their parameters to be estimated relatively easily.

In contrast, John Von Neumann model of an expanding economy allows for choice of techniques, but the coefficient must be estimated for each technology. Wassilly Leontief also introduced optimization problems as to goal equilibrium whether of a household, business, firm or policy maker (Samuela, 2018). It is argued that mathematics allows economists to form meaningful and testable proposition about wide-ranging and complex subjects which could easily be expressed informally. Further, the language of mathematics allows economists to make specific, positive claims about controversial or contentious subjects that would be impossible without mathematics. Much of economics theory is currently presented in terms of mathematical economic models, a set of stylized and simplified mathematical relationships asserted to clarify assumption and implication. This rapid systematizing of economic theories with mathematical principles alarmed the critics of the discipline as well as some notable economists. Notable among these critics are John Marynard Keynes, Dorf & Solon (2016) have criticized the broad use of mathematical models for human behavior, arguing that some human choice is irreducible to mathematics. Despite this argument, the use of mathematical modeling to represent economics theory cannot be suppressed as mathematical tools are the only ways to give meaningful interpretation to economics theories. Precisely, the formal economic modeling began in the l9th century with the use of matrices to represent and explain economic behavior such as utility maximization, an early economic application of mathematical optimization (Chiang, 2015). Economics became more mathematical as a discipline throughout the first half of the 20th century, but the introduction of new and generalized techniques were in the period around the Second World War (Debreel, 2013; Ferrar, 2020; Gass, 2018).

3.0 Methodology of the Study

Dorf & Solon (2016) reported that linear programming is a mathematical method or approach for determining and to achieve the best outcome such as maximum profit in several economics activities such as production projects etc. on the basis of a given optimal criterion (criteria). The word linear is used in describing a proportional relationship between two or more variable in modeling and the word programming means planning of economics activities to get option solution to problems using limited resources by adoption a particular strategy among various strategies to achieve the desired objective (Tucker, 2013; Vanan, 2017). Matrix algebra has proved its effectiveness in analyzing problems that concern the input and output of an economics system.

3.1 Linear Programming as a Case Study

Supposing a firm uses fried proportion of $a_1, \ a_2, a_3$ to produce $b_1, \ b_2, b_3$, then the ratio of its inputs to outputs are given as X_{ij}. The firm’s good of minimizing can be stated as a linear programming problem. For instance,

Maximize $Q = x_1 b_1 + x_2 b_2$ ---------------(1)

Subject to:

$$x_{11}b_1 + x_{12}b_2 + x_{13}b_3 \leq a_1$$
$$x_{21}b_1 + x_{22}b_2 + x_{23}b_3 \leq a_2 \text{ ---------------(2)}$$
$$x_{31}b_1 + x_{32}b_2 + x_{33}b_3 \leq a_3$$
$$b_1 b_2 b_3 \geq 0 \text{ -----------------------(3)}$$

$x_1, \ x_2, x_3$, are the output coefficients for $b_1, \ b_2, b_3$, in total output respectively. Equation (1) is the objective function requiring total output Q to be maximized with respect to $b_1, \ b_2, b_3 \ \geq 0,$ equation (2) and (3) are constraints on output maximization. The first inequality in (2) states clearly that the demand for input (1) for producing ($X_{11}, \ b_1,$) plus the demand for input to produce good 2 ($X_{12}, \ b_2,$) plus the demand for input to produce goods 3($X_{13}, \ b_3,$) should not exceed the supply for a_2 (input 2). Inequality (3) requires that the outputs of good $b_1, \ b_2, b_3,$ should be positive or zero. This is called positive constraints, which rules out intermediate production.

In matrix analysis 1 -3 becomes

$$Q = x_1,\ x_2, x_3 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} ------------ (4)$$

Maximize
Subject to:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \leq \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} ------------ (5) \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ---- (6)$$

These problems can be solved for $b_1,\ b_2, b_3$, by the simplex method while the primal problem involves the maximization of objective functions, the dual problem deals with the maximization of the objective function. As in the problem above, the primal deals with the maximization of the production while the dual deals with notation is of the form.

$$C = [x_1,\ x_2, x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} -------------- (7)$$

Maximize

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{21} & x_{23} \end{bmatrix} -------------- (8)$$

Subject to:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq 0 -------------- (9)$$

The solution for $y_1,\ y_2, y_3$ will minimize the firm's cost of production. There are two methods of handling linear programming problems. They are: The Simplex method and the graphical method.

3.1.1 Solving Linear Programming Problems Using Simplex Method: The simplex method goes through the basic feasible solution in such a way that value Z, of the linear function to be minimize is decreased by each choice of basic feasible solution. A basic feasible solution of linear equation always has the number of non-error's equal number of equations provided the system is non-degenerate.
3.1.2 Application of Linear Programming Method: There is a wide variety of problems to which linear programming method and procedure have been successfully adopted or applied which are; The diet problem is to determine the minimum requirement of nutrients subject to availability of foods and their prices. The production problems are to decide the production schedule and to satisfy demand and to minimize costs in the face of fluctuating rates and storage expenses. While purchasing problems is to have the lease cost objective in, say, the processing of goods purchased from outside and varying in quantity, quality and prices.
3.1.3 Feasible Solution: A solution that satisfies all the constraints of the linear programming problem is called feasible solution. If a feasible solution gives optimum value of the objective function, then it is called optimum solution.
3.1.4 Constraints: A constraint is a condition that a solution to an optimization must satisfy.
3.1.5 Basic Solution: If there are M consistent and independent constraint equation or inequalities and N non-negative variables, we have (n-m) variables to the equal to zero and m basic variable for which there is a unique solution subject to constraints.

3.1.6 Standard Form of a Linear Programming Problem: If the simplex method is to be employ in attacking a linear programming problem, then the problem must first be converted into standard form. This standard form must constitute the following: A constraint should be expressed as equation (this is achieved by adding slack, surplus or artificial variable. The right hand side of the constraints should be made non-negative (if originally it is negative, multiply by (-1). Example: Solve the following programming problem

Maximize $Z = a_1x_1 + a_2x_2 + \cdots \ldots + a_nx_n$

$$b_1x_1 + b_{12}x_{12} + b_{13}x_{13} + \cdots . b_{1n}x_n \geq C_1$$
$$b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + \cdots . b_{2n}x_n \geq C_2$$
$$b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + \cdots . b_{3n}x_n \geq C_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$b_{m1}x_1 \;\; b_{m2}x_2 \;\; b_{m3}x_3 + \cdots + b_{mn}x_n \leq -C_m$$

Transforming the system into standard form before solving the linear programming problem, we have: Maximize $Z = Z = a_1x_1 + a_2x_2 + \cdots \ldots + a_nx_n$

Subject to

$$b_{11}x_1 + b_{12}x_2 + \cdots + b_{in}x_n + A_1 = C_1$$
$$b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n + S_2 + A_2 = C_2$$
$$b_{31}x_1 + b_{32}x_2 + \cdots + b_{2n}x_2 + S_3 + A_3 = C_3$$
$$b_{m1}x_1 + b_{m2}x_2 + \cdots + b_{mn}x_n + A_m = C_m$$

3.1.7 Test for Optimality: Calculate $C_j - Z_j$ for all variables to obtain Z_j multiply each entry in the column C_β by their corresponding entry in column j and sum them i.e.

$$Z_j = \sum C_{\beta i}a_j$$

Where a_j is the column vector in column j.

Examine $C_j - Z_j$ two cases may arise and they are

Case I

If all $C_j - Z_j \leq 0,$ then the basic feasible solution is optimal

Case II

If $C_j - Z_j > 0$ for at least one column, select $C_k - Z_k$ to be maximize over j for $C_j - Z_j > 0$

Select $C_k - Z_k$ to maximize if positive I & II case will arise here

Case I: If $a_k \leq 0,$ then we have an unbounded solution.

Case II: If at least one entry is $a_k > 0$ then an improvement is possible.

3.1.8 Entering Basic Feasible Solution: If case II holds then select the column that has the maximum $C_j - Z_j$ to be the entering basic feasible solution and that column is called a pivot column.

3.1.9 Leaving Basic Feasible Solution: To test for feasibility after identifying the basic feasible solution, the next thing is to determine the variable to leave the basis. For this, we have to divide $X\beta r$ by the corresponding positive row element in the column. $\frac{X\beta r}{ark}$ and select the ratio i.e. the minimum ratio among the obtained ration $\frac{X\beta r}{ark} = \frac{Min}{ak>0}\left[\frac{X\beta r}{ark}\right]$, so that ark is the pivot element. For example, the dependable company plans production of three of their products A, B, and C. The unit products are 2N, N3, and N1 respectively and they require two resource labor and materials. The company operation research department formulates the following linear programming model for deterring the optimal production mix.

Max: $Z = 2p + 3q + r$

Such that:

$$1/3p + 1/3q + 1/3r \leq 1(\text{Labour})$$
$$1/3p + 4/3q + 7/3r \leq 3(\text{Material})$$

Where *p*, *q* and *r* are the number of product A, A and C produced.

Solution
Putting the above problem in standard form, we have:

$$Max{:}Z = 2p + 3q + r + 0s + 0t$$

Such that:

$$1/3p + 1/3q + 1/3r + S = 1$$
$$1/3p + 4/3q + 7/3r + t = 3$$

Table 1:

	CB	Basis	P	Q	r	s	t	b	Φ
R_1	0	s	1/3	1/3	1/3	1	0	1	3
R_2	0	t	1/3	4/3	7/3	0	1	3	9/4
		Cj	2	3	1	0	0	Z=0	

Note Cj row must be zero or negative for maximization problem. Any variable that does not appear in the basis column is considered to be zero.
Cj row is calculated by the formula:

$$C_1 = 2 - (0,0)\begin{bmatrix}1/3\\1/3\end{bmatrix} = 2, C_2 = 3 - (0,0)\begin{bmatrix}1/3\\4/3\end{bmatrix} = 3,$$
$$C_3 = 1 - (0,0)\begin{bmatrix}1/3\\2/3\end{bmatrix} = 1 C_4 = 0 - (0,0)\begin{bmatrix}1\\0\end{bmatrix} = 0$$
$$C_5 = 0 - (0,0)\begin{bmatrix}0\\1\end{bmatrix} = 0,$$

Now, since all the Cj values are not zero or negative for maximization problem, we have not reach the optimal solution.
By considering the highest positive values in Cj row i.e. 3
Now, applying the minimum ratio rule (MRR) and using

$R_4 = 3/4R_4$,

$R_3 = R_1 - 1/3R_4$ we obtain"

Table 2

	CB	Basis	P	q	r	S	t	b	Φ
R_4	0	s	¼	0	-1/4	1	-1/4	¼	1/4÷1/4 =1
R_3	3	q	¼	1	7/4	0	¾	9/4	9/4÷1/4 =10/4
		Cj	5/4	0	-17/4	0	-9/4		

We now test the optimal condition by computing the Cj as follows

$$C_1 = 2 - (0,3)\begin{bmatrix}1/4\\1/4\end{bmatrix} = 2 - \frac{3}{4} = 5/4, C_2 = 3 - (0,3)\begin{bmatrix}0\\1\end{bmatrix} = 0,$$
$$C_3 = 1 - (0,3)\begin{bmatrix}-1/4\\7/4\end{bmatrix} = -17/4 C_4 = 0 - (0,3)\begin{bmatrix}-1/4\\3/4\end{bmatrix} = -9/4$$

Since there is positive value in the Cj, we have not reach the optimal solution, we thus choose the highest positive value and apply the minimum ration rule (MRR) to the table 2

By using $R_5 = 4R_3 \; and \; R_6 = R_4 - 1/4R_5$
We obtain:

Table 3

	CB	Basis	p	q	r	s	t	b
R_5	2	P	1	0	-1	4	-1	1
R_6	3	T	0	1	2	-1	1	2 Z=8
		Cj	0	0	-3	-5	-1	

To test for the optional condition, we compute Cj as follows

$$C_1 = 2 - (2,3)\begin{bmatrix}1\\0\end{bmatrix} = 0, C_2 = 3 - (2,3)\begin{bmatrix}1\\0\end{bmatrix} = 0,$$

$$C_3 = 1 - (2,3)\begin{bmatrix}-1\\2\end{bmatrix} = -3, C_4 = 0 - (2,3)\begin{bmatrix}4\\-1\end{bmatrix} = -5$$

$$C_5 = 0 - (2,3)\begin{bmatrix}-1\\1\end{bmatrix} = -1$$

Since all the Cj $\leq$ 0, we have reach the optimal solution p=1, q=2 and r=0

Interpretation: 1 unit of product A and 2 unit of product B can be produced to make a maximum product of N8.

3.2 Solving Linear Programming Problems using Graphical Method

If the linear programming involves two variables, simple graph is often the most efficient method. The following are steps to be taken while solving a linear programming graphically. The collection of all feasible solution to the linear programming constitute a convex set whose extreme point correspond to the basis solution of feasibility. There are finite numbers of basic solutions within the feasible solution space. At least one of the extreme points gives the optimal solution. If there are two or more optimal solutions, then the convex contribution of the two or more solution is also an optimal solution. The situation where by the feasible region is unbounded so that the optimal value of the objective function cannot be determined means that there is not solution. If it is not possible to find the feasible region which satisfies all the constraints, then the linear programming is said to have an infeasible solution. Example: A firm can produce goods either by a labor intensive technique, using 8 unites of labor and 1 unit of capital or a capital intensive technique using 1 unit of labor and 2 of capital. The firm can arrange up to 200 units of labor and 100 units of capital. It can sell the goods at a constant price *P*. Solution: P is obtained after subtracting cost, so obviously we have simplified the problem because in this way, p becomes the profit per unit. Let p =1, let X_1 and X_2 be the quantities of the goods produced by the process 1 and 2 respectively. To minimize the profit PX_1+PX_2 (since P=1) the problem becomes

Max Z= $X_1 + X_2$

Subject to

$8X_1 + X \leq 200$

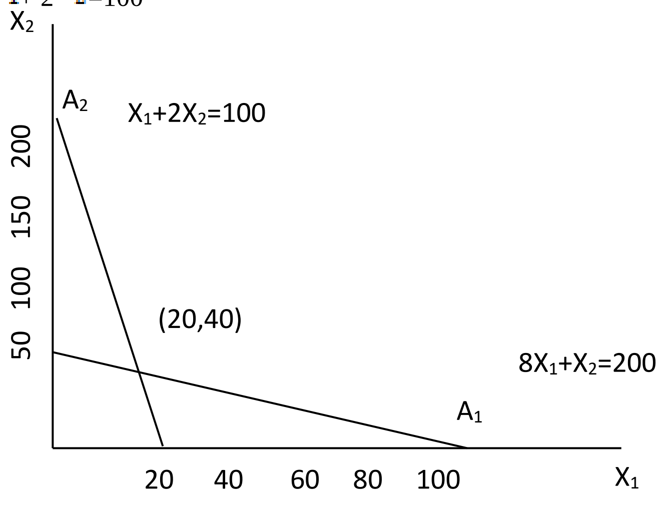
$X_1 + 2X_2 \leq 100$

$\underline{X} \leq 0$

This is a problem in linear programming and since there are two variables involved, we can easily use the graphical approach to arrive at the solution. The procedure is first to draw the graph of the linear constraints treating them as equation.

$8X_1 + X_2 = 200$

$X_1 + 2X_2 = 100$



We have to satisfy simultaneously two constraints of the type $\leq$. The acceptable points are on the south – west axis. Each pair of values (X_1+X_2) in the region is a feasible solution. The feasible region includes all the points inside the region and even those on the boundary lines.

The optimal feasible solution: X_1=20, X_2=40
Then Z = X_1 + X_2
20 + 40 = 60
A solution of 20 units of the goods produced by the first techniques and 40 by the second technique gives the maximum profit of 60,

3.2.1 Jacobian Method: A Jacobian determinant permits testing for functional dependency both linear and non-linear programming. A Jacobian determining /j/ composed of ll the first order partial derivatives of a system of equations arrange in ordered sequence for instance.

Given

$$a_1 = f_1(x_1x_2x_3)$$
$$a_2 = f_2(x_1x_2x_3)$$
$$a_3 = f_3(x_1x_2x_3)$$
$$/j/= \left|\frac{\partial(x_1x_2x_3)}{\partial(x_1x_2x_3)}\right|$$

Notice that the elements of each row are the partial derivatives of one functional Yi with respect to each of the independent variable x_1, x_2, x_3 and the elements of the functions a_1, a_2, a_3 with respect to one of the functional dependent, if /j/ = 0, then the equation are functionally independent

$$\begin{vmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{vmatrix}$$

For instance,

$$Let\ a_1 = 5x_1 + 3x_2$$
$$a_2 = 25x_1^2 + 3x_1x_2 + 9x_1^2$$

Use the Jacobian method of transformation to test for functional dependent.
Solution
First, take the first order partial derivation

$$\frac{\partial a_1}{\partial x_1} = 5$$
$$\frac{\partial a_1}{\partial x_2} = 3$$
$$\frac{\partial a_2}{\partial x_1} = 50x_1 + 30x_2,$$
$$\frac{\partial a_2}{\partial x_1} = 30x_1 + 18x_2$$

Then setting up the Jacobian, we have

$$/j/= \begin{vmatrix} 5 & 3 \\ 50x_1 + 30x_2 & 30x_1 + 18x_2 \end{vmatrix}$$

Evaluate

$$/j/= 5(30x_1 + 18x_2) - 3(50x_1 + 30x_2)$$
$$= 150x_1 + 90x_2 - 150x_1 + 90x_2$$

Since /*J*/ = 0, there is functional dependence between the equations in this simplex cases:

$$(5x_1 + 3x_2) = 25x_1^2 + 30x_1x_2 + 9x_2^2$$

3.2.2 Characteristics Roots and Vectors (Eigen Values, Eigen Vectors: The characteristics roots of a matrix are used to test sign definiteness. Given a square matrix A, it is possible to find a vector V≠0 and a scalar C such that AV= CV. The scalar C is called the characteristics roots or eigen value and the vector V is called the characteristic vector, latent vector or eigen vector. The above equation can also be expressed thus

$$AV = \frac{C}{V}$$

Which can be arranged so that

$$AV - \frac{C}{V} = 0$$
$$(a - Cl)V = 0$$

Where $A - Cl$ is called the characteristics matrix of A since by assumption V≠0, the characteristics matrix A – Cl must be singular and thus its determinant must vanish.

$If\ A = 3 \times 3$ matrix, then

$$/A - Cl/= \begin{vmatrix} a_{11-c} & a_{12} & a_{13} \\ a_{21} & a_{22-c} & a_{23} \\ a_{31} & a_{32} & a_{33} - c \end{vmatrix} = 0$$

With $/A - Cl/= 0,$ there will be an infinite number of solution. To force a unique solution, the solution may be normalized by requiring of the element V_1 of V that V_{12} = 1. If all C's are positive, then A is positive definite. If they are negative, then A is negative definite. If they are positive and at least one A is negative definite. If they are positive and at least one C=0 then A is positive semi-definite. If they are negative and at least once =0, then A is positive semi-definite, if they are negative and at least one =0, the A is negative semi-definite, if some of them are positive and others negative, then A is semi-indefinite. Example

$$Given\ A = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

Find the characteristic roots of A and the characteristics vector

Solution

To find the characteristics root of A, the determinant of the characteristic matrix i.e A- Cl must be zero.

$$/A = Cl/= \begin{vmatrix} -2 - c & 4 \\ 4 & -2 - c \end{vmatrix} = 0$$
$$(-2 - c)(-2 - c) - (4)(4) = 0$$
$$C^2 + 4C - 12 = 0$$
$$(C - 6)(C + 2)$$

C_1 = +6 or C_2 = -2

Testing for sign definition sine once characteristics root is negative and the other is positive, then A is indefinite.

NOTE:

1. $C_1 + C_2$ must be equal to the sum of the element on the principal diagonal of A
2. C_1C_2 must be equal to the determinant of the original matrix A.

   Now to find the characteristics vector, we substitute the first characteristics root C_1 = -6

In $\begin{bmatrix} -2+6 & 4 \\ 4 & -2+6 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Since the coefficient matrix is linearly dependent, then there is the finite number of solution. The production of the matrix gives two equations which are identical.

$4V_1 + 4V_2 = 0$ (by multiplying the 2 x2 matrix by the column vector solving V_2 in terms of V_1) $V_2 = -V_1$
Then, normalizing the solution so that
$V_1^2 + (-V_2^2) = 1$
$V_2 = -V_1$ is substituted in the equation above to get
$V_1^2 + (-V_2^2) = 1$
Thus $2V_1^2 = 1 \rightarrow V_1^2 = 1/2$
Taking the positive square root of V_1 since $V_2 = +V_1$
$V_2 = -V_1 = -\sqrt{0.5}$
Hence, the first characteristic vector is

$$V_1 = \begin{bmatrix} \sqrt{0.5} \\ -\sqrt{0.5} \end{bmatrix}$$

When the second characteristics root $C_2 = -2$ is used to get

$$\begin{bmatrix} -2+6 & 4 \\ 4 & -2+6 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Multiply through by bij: $-4V_1 + 4V_2 =$
Thus $V_1 = V_2$ *(By normalizing)*
$V_1^2 + V_2^2 = 1$
Since $V_1 = V_2$ then
$V_1^2 + V_2^2 = 1$

$$2V_2^2 = 1 \rightarrow V_2^2 = \frac{1}{2} \rightarrow V_2 = \sqrt{0.5}$$

Hence
$V_2 = \sqrt{0.5}\ and\ V_1 = \sqrt{0.5}$
Thus

$$V_1 = \begin{bmatrix} \sqrt{0.5} \\ \sqrt{0.5} \end{bmatrix}$$

3.2.3 Determinant: Determinant may be used to test for positive or negative definite of any quadratic form. The determinant of a quadratic form is called a discriminant /D/
$Z = ax^2 + bxy + cy^2$
The discriminant is found by placing the coefficient of the square matrix on the principal diagonal and dividing the coefficient of the non-squared term equally between the off diagonal positions.

$$/D/= \begin{vmatrix} a & b/2 \\ b/2 & a \end{vmatrix}$$

Then evaluating the principal minors

$$/D/= a \; and /D_2/= \begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = ac - {b^2}/_4$$

If $/D_1/,/D_2/>/D/$ is positive definite and Z is positive for all non zero values of *x* and *y*. if $/D/< 0,Z$ is negative definite and is negative for sign definite and Z may assume both positive and negative values.

Example

Given the quadratic equation $Z = 2x^2 + 5xy + 8y^2$

$$/D/= \begin{vmatrix} 2 & 2.5 \\ 2.5 & 8 \end{vmatrix}$$

To evaluate the principal minors

$$/D_2/= 2 > 0$$

$$D_2 = \begin{vmatrix} 2 & 2.5 \\ 2.5 & 8 \end{vmatrix} = 16 - 6.25 = 9.75 > 0$$

Thus, Z is positive definite, meaning that it will be greater than zero for all non-zero values of *x* and *y*.

3.3.4 Input – Output Analysis: In a modern economy where the production of one goods require the input of many other goods and intermediate goods in production process for instance steel require coal, iron, ore, electricity etc, total demand X for produce one will be the summation of all intermediate demand for the product plus the final demand for the product amongst the consumers, investors, the government and exporters, as ultimate users, one a_{ij} is a technical coefficient expressing the value of input one required to produce one naira worth of product, the total demand for product one can be expressed as $X_i = a_{ij}X_1 + a_{ij}X_2 \dots \dots + a_{in}X_n + b_i$ for 1 = 1, 2, ….., n in the matrix form, this can be expressed as $X = Au + B$

Where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} and \; A = \begin{bmatrix} 1 - a_{11} & -a_{12} & \dots \dots \dots & a_{13} \\ -a_{21} & 1 - a_{22} & \dots \dots \dots & a_{23} \\ -a_{n1} & -a_{n2} & \dots \dots \dots & 1 - a_{nn} \end{bmatrix}$$

"A" is called the matrix of technical coefficient. To find the level of total output (intermediate and final) needed to satisfy final demand we can solve for 'x' in terms of the matrix of technical coefficient and column vector of final demand, both of which are given x –Ax = B.

$$(1 - A)x = B$$

$$x = (1 - A)^{-1}B$$

Thus, for a three sector economy we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 - a_{11} & -a_{12} & \dots \dots \dots & a_{13} \\ -a_{21} & 1 - a_{22} & \dots \dots \dots & a_{23} \\ -a_{n1} & -a_{n2} & \dots \dots \dots & 1 - a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Where 1-A matrix is called Leontief matrix in a complete input-output table, labour and capital would also be included as inputs consisting value added by the firm. The vertical summation of elements along column j in such a model would equal one and the input cost of producing of producing the unit or one naira's worth of the commodity.

Example
Determine the total demand x for industries 1,2 and 3 given the matrix of technical coefficient A and the final demand Vector B.

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.1 \\ 0.3 & 0.2 & 0.3 \end{bmatrix} \; and \; B = \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix}$$

Taking the inverse to obtain:

$$(1-A)^{-1} = \begin{bmatrix} 1.43 & -2 & -2.5 \\ -10 & 1.7 & -10 \\ -3.3 & -5 & 1.43 \end{bmatrix}$$

$$\begin{bmatrix} 1.43 & -2 & -2.5 \\ -10 & 1.7 & -10 \\ -3.3 & -5 & 1.43 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix} = \begin{bmatrix} -91.4 \\ -583.0 \\ 108.8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4.0 Matrix Application in Economics
Let assume that an economic system has in different industries say 11, 12, 13 ….m, in which each of them need input like (raw materials, utilities) and output like finished product. The input coefficient b_{ij}measures the amount of input *jth* industry needs from the *ith* industry to produce one unit. The collection of input coefficient is given by the following m x m matrix 12, 13…. lm (users).

$$D = \begin{bmatrix} b_{11} & b_{12} & b_{1m} \\ b_{21} & b_{22} & b_{2m} \\ b_{m1} & b_{m2} & b_{mm} \end{bmatrix}$$

In which 11, 12, 13 ….m (suppliers), this matrix is called the input – output matrix. In getting how to use the matrix better, lets imagine that the entries of D are given in naira. The total amount spent by the *jth* industry to produce one-naira worth of output is given by the sum of the entries in the *jth*column. Thus for a giving matrix to be an input –output matrix, it must have the following characteristics:

1. The matrix must be a square matrix of the m x m
2. Each entry must be a non –negative and less than 1
3. The sum of the entries of any column must be less than 1 i.e $b_{ij} + b_{2i} + b_{3i} \dots\dots\dots b_{mj} \le 1$

4.1 Open and Closed System
In the open economical system, stocks are normally bought and sold in the open market, while in a closed economic system, stocks are being sold only to industries within the system. Whereas, if the economic system is closed, then the total output of the kth industry is given by linear equation.

$$y_1 = b_{11}y_1 + b_{12}y_2 + \cdots \dots\dots + b_{1m}y_m$$
$$y_2 = b_{21}y_1 + b_{22}y_2 + \cdots \dots\dots + b_{2m}y_m$$
$$y_m = b_{m1}y_1 + b_{m2}y_2 + \cdots \dots\dots + b_{mm}y_m$$

For the closed economical system, the sum of the entries in each column should be one i.e

$$b_{ij} + b_{2j} + b_{3j} \dots\dots\dots b_{mj} = 1$$

The matrix of the system Y=DY, where Y is the output of the matrix. In an open system where product isbeing sold to group outside the system, the total output of the kth industry is given by

$Y_1 = b_{K1}Y_1 + b_{K2}Y_2 \dots\dots\dots b_{km}Y_m + e_K$ where e_K represent the external demand for the kth industry's products.

Now the collection of the total output for an open system is therefore represented by the following system of linear equation.

$$y_1 = b_{11}y_1 + b_{12}y_2 + \cdots \dots\dots + b_{1m}y_m + e_1$$
$$y_2 = b_{21}y_1 + b_{22}y_2 + \cdots \dots\dots + b_{2m}y_m + e_2$$
$$y_m = b_{m1}y_1 + b_{m2}y_2 + \cdots \dots\dots + b_{mm}y_m + e_m$$

The matrix form of this system is $Y = Dy + E$ where y is the output matrix and E is the external demand matrix. Example

1. A system has three industries with the following output: to produce one-naira output, the first industry requires the following

N0.10 of the 1st industry Product
N0.15 of the 2nd industry product
N0.23 of the 3rd industry product

2. To produce one-pound worth of output 2nd industry the following is required.

N0.43 of the 1st industry Product
N0.00 of the 2nd industry product
N0.03 of the 3rd industry product

3. To produce one-pound worth of output 3rd industry requires the following

N0.00 of the 1st industry Product
N0.37 of the 2nd industry product
N0.02 of the 3rd industry product

$$D = \begin{bmatrix} 0.10 & 0.43 & 0.00 \\ 0.15 & 0.00 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{bmatrix}$$

4.2 Finding the Output Matrix of an Open Economics System

Using the above example to solve for the output matrix y with equation y= Dy + E, where the external demand is given as

$$\begin{bmatrix} 20 & 000 \\ 20 & 000 \\ 25 & 000 \end{bmatrix}$$

Solution

Letting one be 1th identity matrix, we can write the equation

$X = D_X + E$ as

$1_X = D_X + E,$ this means that

$1_X - D_X = E,$

$= (1 - 0)_X = E,$

Using D in the example above

$$1 - D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.10 & 0.43 & 0.00 \\ 0.15 & 0.00 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{bmatrix} = \begin{bmatrix} 0.90 & -0.43 & 0.00 \\ -0.15 & 1.00 & -0.37 \\ -0.23 & -0.03 & 0.93 \end{bmatrix}$$

Apply Gauss –Jordan elimination approach to the system of linear equation represented by

$(1_X - D)_X = E,$ it produces

$$\begin{bmatrix} 0.90 & -0.43 & 0.00 & 20{,}000 \\ -0.15 & 1.00 & -0.37 & 30{,}000 \\ -0.23 & -0.03 & 0.93 & 25{,}000 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 22 \\ 0 & 1 & 22 \\ 0 & 0 & 22 \end{bmatrix}$$

Therefore, the output matrix is

$$x = \begin{bmatrix} 46 & 616 \\ 51 & 058 \\ 38 & 014 \end{bmatrix}$$

And we calculate that the outputs for the three industries are as follows:

Output for 1st industry: 22,110 units

Output for 2nd industry: 25,058 units
Output for 3rd industry: 10,1014 units

4.3 Solving for the Output Matrix of a Closed Economic System

The closed economical system is made up of four different industries, each of which uses produce form the other three industries as shown below table 4

.

Table 4:

Supplier	Industry 1	Industry 2	Industry 3	Industry 4
Industry 1	0.15	0.25	0.35	0.05
Industry 2	0.30	0.15	0.20	0.40
Industry 3	0.35	0.40	0.30	0.20
Industry 4	0.20	0.20	0.15	0.30

If industry 4 produces 17,730 units in a year, how may units do the other three industries produce?

Solution

Observe that the sum of each column in the above matrix is one which satisfies the condition for the entries in a closed economic system thus, the input-output matrix system of linear equation representing this system is as follows:

$X_1 = 0.15x_1 + 0.25x_2 + 0.35x_3 + 0.05x_4$
$X_2 = 0.03x_1 + 0.15x_2 + 0.20x_3 + 0.40x_4$
$X_3 = 0.35x_1 + 0.40x_2 + 0.30x_3 + 0.20x_4$
$X_4 = 0.20x_1 + 0.20x_2 + 0.15x_3 + 0.30x_4$

In standard form i.e (1-x)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0.15 & 0.25 & 0.35 & 0.05 \\ 0.30 & 0.15 & 0.20 & 0.40 \\ 0.35 & 0.40 & 0.30 & 0.20 \end{bmatrix}$$

This system becomes

$0.85x_1 - 0.25x_2 + 0.35x_3 + 0.050x_4 = 0$
$-0.30x_1 - 0.85x_2 + 0.70x_3 + 0.40x_4 = 0$
$-0.35x_1 - 0.40x_2 + 0.70x_3 + 0.20x_4 = 0$
$0.20x_1 - 0.20x_2 + 0.15x_3 + 0.70x_4 = 0$

Finally, using the Gauss-Jordan elimination method, we have

$X_1 = /909x_4/1773$
$X_2 = /271x_4/1773$ and
$X_3 = /2830x_4/1773,$ since $X_4 = 17{,}73$

Then the four industries must produce the following number of limits
Output for industry 1 = 19,090 units
Output for industry 2 = 21,740 units
Output for industry 3 = 28,300 units
Output for industry 4 = 17730 units

5.0 Conclusion of the Study

We have so far discussed matrices used in economic and special determinant and saw how to use simple and graphical method and some other method in solving economic and industrial problems. Lastly, we have been able to find the output matrix of an open economic system and so solve the output matrix of a closed economic system. When trying to interpret a linear optimization problem with more than two variables, it becomes very difficult using the graphical method. However, making use of the simplex method makes the optimization problem very easy to interpret. The simplex method which is now on interactive computation procedure has more advantage over other matrix methods such that, it can be used to solve linear optimization problems whose constraints are inequalities while Crammer's rule and Jacobian's method are used in solving systems of linear equations. Above all, using the simplex method makes it easier to interpret the result as said earlier. Also an economist needs to makes use of matrix in work out economic theories. An economist makes us of equation to represent the relationship between economics variables. An economics draw graphs to illustrate the nature of this relationship. An economist equally use matrix to solve system of demand and equations to find the equilibrium price of goods. Through my research work, emphasis has been placed on the application of matrices in economics. Using matrix, economists studying the economy of the nation should be able to study the equilibrium graph to ascertain when a commodity is needed in the economy for supply to consumers. I am therefore recommending that matrix should be use by the mathematicians to go into the economy of the nation.

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